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Superpotentials from flux compactifications of M-theory¹

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ABSTRACT

In flux compactifications of M-theory a superpotential is generated whose explicit form depends on the structure group of the 7-dimensional internal manifold. In this note, we discuss superpotentials for the structure groups: G_2 , $SU(3)$ or $SU(2)$. For the G_2 case all internal fluxes have to vanish. For $SU(3)$ structures, the non-zero flux components entering the superpotential describe an effective 1-dimensional model and a Chern-Simons model if there are $SU(2)$ structures.

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In order to make phenomenological predictions one has to fix the moduli appearing in string or M-theory compactifications. One way of doing this is to consider flux compactifications, where the generated superpotential might fix most of the moduli, although it is not clear whether all moduli can be fixed. The resulting potentials have not only supersymmetric extrema, related to anti deSitter (AdS) or preferable flat space vacua, but may also have de Sitter vacua which are interesting in its own.

From the Killing spinor equations we get not only constraints on the fluxes and the geometry of the internal space, see [1, 2, 3], but also, as we will present in this note, a procedure to calculate the superpotential. In the usual way one derives the superpotential by dimensional reduction or from calibrated submanifolds [4, 5, 6], but we will follow another route. We introduce the superpotential as mass term for the 4-d gravitino(s), implying that the four-dimensional (4-d) Killing spinor is not covariantly constant, and hence the 11-d Killing spinor equations relate directly the lower-dimensional superpotential to the fluxes.

By definition, in the vacuum all 4-d scalars as well as gauge fields are trivial and hence the metric is either flat or AdS. We write therefore the 11-d bosonic fields as

$$ds^2 = e^{2A} \left[g_{\mu\nu}^{(4)} dx^\mu dx^\nu + h_{ab} dy^a dy^b \right], \quad (1)$$

$$F = m dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 + \frac{1}{4!} F_{abcd} dy^a \wedge dy^b \wedge dy^c \wedge dy^d$$

where $A = A(y)$ is a function of the coordinates of the internal 7-manifold X_7 with the metric h_{ab} and we denote the Freund-Rubin parameter as m and $g_{\mu\nu}^{(4)}$ is the 4-d metric. Unfortunately, we have to omit here many technical details which are important for the understanding and refer only to the literature [3].

Unbroken supersymmetry requires the existence of (at least) one Killing spinor η yielding a vanishing gravitino variation of 11-dimensional supergravity

$$\begin{aligned} 0 = \delta\Psi_M &= \left[\partial_M + \frac{1}{4} \hat{\omega}_M^{RS} \Gamma_{RS} + \frac{1}{144} \left(\Gamma_M^{NPQR} - 8 \delta_M^N \Gamma^{PQR} \right) F_{NPQR} \right] \eta \\ &= \left[\partial_M + \frac{1}{4} \hat{\omega}_M^{RS} \Gamma_{RS} + \frac{1}{144} \left(\Gamma_M \hat{F} - 12 \hat{F}_M \right) \right] \eta. \end{aligned} \quad (2)$$

In the second line we used the formula: $\Gamma_M \Gamma^{N_1 \dots N_n} = \Gamma_M^{N_1 \dots N_n} + n \delta_M^{[N_1} \Gamma^{N_2 \dots N_n]}$ and introduced the abbreviation $\hat{F} \equiv F_{MNPQ} \Gamma^{MNPQ}$, $\hat{F}_M \equiv F_{MNPQ} \Gamma^{NPQ}$. We decompose

the Γ -matrices as usual $\Gamma^\mu = \hat{\gamma}^\mu \otimes 1$, $\Gamma^{a+3} = \hat{\gamma}^5 \otimes \gamma^a$ with $\mu = 0, 1, 2, 3$, $a = 1, 2, \dots, 7$ and we find for the field strength

$$\hat{F} = -i m \hat{\gamma}^5 \otimes \mathbf{1} + \mathbf{1} \otimes F, \quad \hat{F}_\mu = \frac{1}{4} i m \hat{\gamma}^5 \hat{\gamma}_\mu \otimes \mathbf{1}, \quad \hat{F}_a = \hat{\gamma}^5 \otimes F_a. \quad (3)$$

One distinguishes the external and internal variation, where the internal variation gives a differential equation fixing not only the spinor but giving also differential equations for the metric (or the vielbeine). Its variation reads

$$0 = \left[\mathbf{1} \otimes \left(\nabla_a^{(h)} + \frac{1}{2} \gamma_a{}^b \partial_b A + \frac{i m}{144} \gamma_a \right) + \frac{1}{144} e^{-3A} \hat{\gamma}^5 \otimes \left(\gamma_a F - 12 F_a \right) \right] \eta. \quad (4)$$

In this paper we will explore especially the external variation which becomes

$$0 = \left[\nabla_\mu \otimes 1 + \hat{\gamma}_\mu \hat{\gamma}^5 \otimes \left(\frac{1}{2} \partial A + \frac{i m}{36} \right) + \frac{1}{144} e^{-3A} \hat{\gamma}_\mu \otimes F \right] \eta \quad (5)$$

where $\partial A \equiv \gamma^a \partial_a A$ and ∇_μ is the 4-d covariant derivative in the metric $g_{\mu\nu}^{(4)}$.

As next step one has to expand the 11-d Majorana spinor in all independent 4-d spinors ϵ^i and 7-d (real) spinors θ_i giving

$$\eta = \sum_{i=1}^N \epsilon^i \otimes \theta_i \equiv \epsilon^i \otimes \theta_i. \quad (6)$$

If there are no fluxes, all of these spinors are covariantly constant and N gives the number of extended supersymmetries in 4 dimensions, with $N = 8$ as the maximal supersymmetric case. If one turns on fluxes, not all spinors are independent resulting in a reduction of the number of preserved supersymmetries. In the least supersymmetric case, we have only a single spinor on X_7 , which at each point of the internal space can be written as a singlet of $G_2 \subset SO(7)$, but in general the embedding changes from point to point. In fact the corresponding rotation is related to the structure group $G \subset G_2$ and the spinor is only a singlet under the structure group, but in general not under G_2 . One expands now the 11-d spinor η with respect to the maximal number of singlets under the structure group, i.e. for G_2 : $N=1$; $SU(3)$: $N=2$; $Sp(2)$: $N=3$ and for $SU(2)$: $N=4$. Recall, only for very specific fluxes this agrees with the number of supersymmetries and for generic fluxes we will always get constraints on the spinors so that we encounter in general an $N=1$ vacuum.

Before we come to a discussion of the different cases, we have to introduce the superpotential, which by definition is the mass term of the 4-d gravitino(s). This implies

that the 4-d spinor(s) are not covariantly constant but satisfy the equation (for simplicity we put all numeric factors into the superpotential)

$$\nabla_\mu \epsilon^i = \hat{\gamma}_\mu (W_1^{ij} + i\hat{\gamma}^5 W_2^{ij}) \epsilon_j . \quad (7)$$

If N is even one may introduce a symplectic Majorana notation, but in order to cover also the case with a single spinor let us stick for the time being to the real notation. This yields for the external variation

$$0 = \left([W_1^{ij} + i\hat{\gamma}^5 W_2^{ij}] \otimes \mathbf{1} + \delta^{ij} \left[\frac{im}{36} \hat{\gamma}^5 \otimes \mathbf{1} + \frac{1}{2} \hat{\gamma}^5 \otimes \partial A + \frac{1}{144} e^{-3A} (\mathbf{1} \otimes F) \right] \right) \epsilon_i \otimes \theta_j . \quad (8)$$

Note, $W_{1/2}^{ij}$ mixes the different components of ϵ^i and in general one has to impose constraints on the 4-d spinors to solve these equations. Let us now consider the different cases separately.

Case (i): G_2 -structure

If the structure group is the whole G_2 , only one real spinor on X_7 can be singlet. Hence the 11-d spinor is written as

$$\eta = \epsilon \otimes \theta \quad (9)$$

and since the 11- and 7-d spinor are Majorana also the 4-d spinor ϵ has to be Majorana. In this case the spinor $\theta = e^Z \theta_0$ (with the real function Z) is a G_2 singlet, i.e. the constant spinor θ_0 obeys: $\gamma_{ab} \theta_0 = -i\varphi_{abc} \gamma^c \theta_0$, with the G_2 invariant 3-index tensor φ_{abc} which in turn can be defined as fermionic bi-linear $\varphi_{abc} = -i\theta_0^T \gamma_{abc} \theta_0$ (note: $\theta_0^T \gamma_a \theta_0 = \theta_0^T \gamma_{ab} \theta_0 = 0$). In our notation, the 7-d γ -matrices are purely imaginary and the $\theta_0 = (1, 0, 0, \dots)$.

Now, inserting this spinor ansatz into (8) we get two equations; one proportional to ϵ and another proportional to $\hat{\gamma}^5 \epsilon$. Contracting them θ gives

$$W_1 = \frac{1}{144} e^{-3A} (\theta^T F \theta) = \frac{1}{144} e^{-3A+2Z} \psi^{abcd} F_{abcd} , \quad (10)$$

$$W_2 = \frac{m}{36} \quad (11)$$

where the 4-index tensor ψ_{abcd} as the dual of φ_{abc} . This superpotential can be written in the form proposed in [6]: $W \sim \int F \wedge (\varphi + \frac{i}{2} C)$ if one uses the equations of motion for the gauge field: $d(*_{11} F + \frac{1}{2} F \wedge C) = 0$ for the external components, where $\int_{X_7} *_{11} F = \int_{X_7} F_7 \sim \int_{X_7} *_7 m$.

But, since we are dealing with a Majorana spinor, also the internal variation (4) gives two independent equations: one differential equation ($\sim \epsilon$) and another constraints on the fluxes ($\sim \hat{\gamma}^5 \epsilon$). The latter implies that $\psi^{abcd} F_{abcd} = 0$ and hence only the Freund-Rubin parameter $m \sim W_2$ can be non-zero. In fact a detailed analysis shows that all internal fluxes have to be trivial in this case [7, 3].

Case (ii): $SU(3)$ -structure

Next, if we reduce the structure group to $SU(3)$, one can build two singlet spinors on X_7 . These two spinors are equivalent to the existence of a vector field v , which in turn can be expressed as a bi-linear of these two spinors. If we normalize this vector field, these two real spinors can be combined into one complex spinor defined by

$$\theta = \frac{1}{\sqrt{2}} e^Z (1 + v_a \gamma^a) \theta_0 \quad , \quad v_a v^a = 1 \quad (12)$$

where the constant spinor θ_0 is again the G_2 singlet and Z is now a complex function. The vector field defines a foliation of the 7-manifold by a 6-manifold X_6 and both spinors, θ and its complex conjugate θ^* , are chiral spinors on X_6 . The 11-d Majorana spinor is now decomposed as

$$\eta = \epsilon \otimes \theta + \epsilon^* \otimes \theta^* \quad (13)$$

where the complex 4-d spinor is chiral and we choose

$$\hat{\gamma}^5 \epsilon = \epsilon \quad , \quad \hat{\gamma}^5 \epsilon^* = -\epsilon^* \quad .$$

We introduce the superpotential by

$$\nabla_\mu \epsilon = \hat{\gamma}_\mu e^{\frac{K}{2}} \bar{W} \epsilon^*$$

with $\bar{W} = W_1 - i W_2$ and K is the Kähler potential [we use here the known notation from 4-d supergravity]. Due to the opposite chirality, terms with ϵ and ϵ^* are independent and the term $\mathcal{O}(\epsilon)$ of the external variation (8) can be written as

$$e^{\frac{K}{2}} W \theta^* = - \left[\frac{im}{36} + \frac{1}{2} \partial A + \frac{1}{144} e^{-3A} F \right] \theta \quad . \quad (14)$$

Again, we can contract this equation with θ^T and use $\theta^T \theta = \theta^T \gamma_a \theta = 0$ ($\theta^+ \theta = e^{Z+\bar{Z}}$) to find

$$e^{\frac{K}{2}} \bar{W} = \frac{1}{4! 3!} e^{-(3A+Z+\bar{Z})} (\theta^T F \theta) = \frac{i}{36} e^{-3A} v^a F_{abcd} \bar{\Omega}^{bcd} \quad (15)$$

where Ω is a holomorphic 3-form on X_6 . If we define on X_6 the 3-form field strength by $H_{abc} \equiv v^d F_{dabc}$ and if we integrate over the internal space, the superpotential can also be written as (note the volume form on X_6 is: $i\Omega \wedge \bar{\Omega}$)

$$W \sim \int_{X_7} H \wedge \Omega \wedge v \sim \int_{X_7} F \wedge \Omega . \quad (16)$$

Let ω be the associated 2-form to the almost complex structure on X_6 defined by: $\omega_{ab} = \varphi_{abc} v^c$. One finds that: $d\omega \wedge \Omega \sim W i\Omega \wedge \bar{\Omega}$ [3], which allows to express the superpotential also purely geometrically in terms of torsion components. From our setup we cannot distinguish between both expressions, but if v is Killing, we can compare our result with the ones derived in [8] and this suggest that we have to add both expressions yielding

$$W \sim \int_{X_7} (F + iv \wedge d\omega) \wedge \Omega^{(3,0)} . \quad (17)$$

Upon reduction to 10 dimensions, this superpotential contains only fields that are common in all string theories and hence one can make a number of consistency checks (that it is U-dual to the type IIB superpotential and anomaly-free on the heterotic side).

Let us continue and contract eq. (14) with θ^+ and find

$$0 = -\frac{im}{36} + \frac{1}{2} v^a \partial_a A + \frac{1}{144} e^{-3A} (\theta^+ F \theta) .$$

Since the last term is real, we are forced to set $m = 0$ and $v^a \partial_a A = \frac{e^{-3A}}{144} (F_{abcd} \omega^{ab} \omega^{cd})$. If one reduces as first step only over X_6 , the corresponding 5-d superpotential becomes $W_{5d} = (F_{abcd} \omega^{ab} \omega^{cd}) \sim \int_{X_6} F \wedge \omega$. Note, this W_{5d} is compensated by the warp factor A and does not enter the 4-d superpotential. In fact, from the supergravity point of view one obtains in 5 dimensions a domain wall solution and this first order differential equation for A is the known BPS equation, see [5]. The superpotential W on the other hand does not contribute to the 5-d potential, but represents a kinetic term of two (axionic) scalars of a hyper multiplet², which is non-zero in the vacuum and curves the domain wall. That kinetic terms of (axionic) scalars act effectively as potentials (if one allows for a linear dependence of the 5th direction) can be understood from massive T-duality or generalized dimensional reduction [9] and hence we expect that these flux compactifications are related to the curved domain wall appeared in [10]. Let us also note that, because we excluded any dependence on the four external coordinates, this

²These scalars come from the (3,0) and (0,3) part of the 3-form potential C in 11 dimensions.

reduction over X_6 gives an effective 1-dimensional description and if one would take into account singularities giving rise to non-Abelian gauge groups, we expect a matrix model description as discussed in [11].

Case (iii): $SU(2)$ -structure

Strictly speaking the reduction of the structure group would yield as next step the group $SO(5) \simeq Sp(2)$, related to three (real) singlet spinors on X_7 . We will however, not discuss this case here, which for trivial fluxes is related to $N = 3$ models. Instead, we will consider the case where the structure group is $SU(2)$, which allows for four singlet spinors and this case appears as natural continuation of the $SU(3)$ case. It is again natural to work with (two) 4-d chiral spinors and we write the 11-d spinor as

$$\eta = \epsilon^i \otimes \theta_i + cc \quad , \quad \text{with :} \quad \hat{\gamma}^5 \epsilon^i = \epsilon^i \quad (18)$$

where $i = 1, 2$. Similarly we combine the four real spinors on X_7 into two complex spinor defined by

$$\theta_1 = \frac{1}{\sqrt{2}} e^Z (1 + v_a \gamma^a) \theta_0 \quad , \quad \theta_2 = \frac{1}{\sqrt{2}} e^Y (u_a + i w_a) \gamma^a \theta_0 = e^{Y-Z} u_a \gamma^a \theta_1$$

with Z and Y as complex functions and the three vectors are orthogonal and normalized: $|v|^2 = |u|^2 = |w|^2$, $u \cdot w = u \cdot v = w \cdot v = 0$ and obey moreover the relation $w_a = \varphi_{abc} v^b u^c$. In addition, they can be expressed as fermionic bi-linears

$$v_a = e^{-(Z+\bar{Z})} (\theta_1^+ \gamma_a \theta_1) \quad , \quad u_a + i w_a = e^{-(\bar{Z}+Y)} (\theta_1^+ \gamma_a \theta_2) \quad .$$

and to simplify the notation we will in the following set $Y = Z = 0$. These three vectors imply that the 7-manifold X_7 is a fibration of a 3- over a 4-manifold X_4 , which is reminiscent to the known G_2 -manifolds written as R_3 -fibrations over a self-dual Einstein space or an S^3 fibration over a hyper-Kähler space [12, 13]. The concrete geometry is fixed by the differential equations satisfied by these vector fields, which however is not the subject of this paper. Let us continue and note, that the vector v can be used to define an $SU(3)$ structure and the holomorphic vector³ $u + iw$ gives the reduction to $SU(2)$. In this case, the external variation can be written as

$$e^{\frac{K}{2}} W^{ij} \theta_j^* = - \left[\frac{im}{36} + \frac{1}{2} \partial A + \frac{1}{144} e^{-3A} F \right] \theta^i \quad . \quad (19)$$

³We use holomorphicity in a pointwise sense, i.e. using $\varphi_{abc} v^c$ as an almost complex structure on X_6 , we can at each point introduce holomorphic indices and the vector, e.g., obeys: $(1 - i \varphi_{abc} v^c)(u^a + i w^a) = 0$, because $w_a = \varphi_{abc} v^b u^c$. Note, in general X_6 is *not* a complex manifold.

As for the SU(3)-case, we had to collect again terms of the same chirality and the superpotential has been promoted from a complex to an SU(2)-valued quantity, which together with the Kähler potential transforms under the U(2) R-symmetry rotation of the two spinors. It can be written as

$$W^{ij} \equiv W_x(\sigma^x)_k^j \epsilon^{ik} = -i W_x(\sigma^x \cdot \sigma_2)^{ij} \quad (20)$$

where σ^x are the three Pauli matrices (this notation is again borrowed from N=2 gauged supergravity, see [14, 15]) and the three components W_x can be obtained by contracting eq. (19) by θ_i^T . Because $\theta_i^+ \theta_j = \theta_i^T \theta_j^* = \delta_{ij}$ and in addition $\theta_i^T \theta_j = \theta_i^T \gamma_a \theta_j = 0$ we find

$$W_{ij} = \frac{1}{144} e^{-\frac{K}{2} - 3A} (\theta_i^T F \theta_j) . \quad (21)$$

Since $\epsilon^{ij} \theta_i^T F \theta_j \sim \theta_1 \{F, u\} \theta_1 = 0$, this matrix is symmetric which is in agreement with (20). Using the Pauli matrices we can write

$$W \sim \int_{X_7} F \wedge q \wedge \Omega^{(2)} \quad \text{with : } q = u \sigma_3 - iw \mathbf{1} - v \sigma_1 \quad (22)$$

where $\Omega_{ab}^{(2)} = \Omega_{abc}(u^c + iw^c) = 2\Omega_{abc}u^c$ is the holomorphic 2-form on X_4 . In order to ensure that W_x as introduced in (20) are real we impose that the three 2-forms $\hat{F}_2 = [\star(F \wedge v), \star(F \wedge u), \star(F \wedge w)]$ have no components along $\text{Im}\Omega^{(2)}$ on X_4 . Note there are three 2-forms on X_4 which are SU(2) singlets and which one may choose as anti self-dual: two of them are combined in the holomorphic $\Omega^{(2)}$ and the third one is the complex structure. For generic fluxes they might not be exact and similar to the situation with SU(3) structures, we expect here additional contributions to the superpotential, which have not yet been worked out.

Also in this case, the superpotential can effectively be described by a lower dimensional model. We can again ignore the four external coordinates and from the fact that the components of F that enter W have always two legs inside X_4 , we obtain by a dimensional reduction over X_4 an effective Chern-Simons model as discussed in [16, 11]. This Chern-Simons model lives on the 3-manifold identified by the three vectors (v, u, w) , which becomes the worldvolume of D6-branes if X_4 has NUT fixed points (i.e. ADE-type singularities).

To summarize, we derived superpotentials from flux compactifications if the structure group of the internal manifold is G_2 , SU(3) or SU(2) and all three cases yield generically

$N=1$ vacua in four dimensions. The reduction of the structure group was related to additional spinors on X_7 , which in turn implied specific vector fields specifying the superpotential. For $SU(3)$ it is a single vector field, which gives a foliation of X_7 by a 6-manifold X_6 and on this 6-manifold one can define a holomorphic 3-form that enters the superpotential. If the structure group is only $SU(2)$, three vector fields⁴ define a fibration of a 3-space over a 4-manifold X_4 . To get contact with the expressions discussed in [16, 11], one can reduce the $SU(3)$ case over X_6 to obtain an effective 1-d matrix model description and for the $SU(2)$ case a reduction over X_4 would give rise to a 3-d Chern-Simons model. This is an interesting observation which requires however further investigations.

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⁴Any 7-dimensional spin manifold has three vector fields [17] and hence one can always define $SU(2)$ structures.

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